Problem Set #3, solutions

Problem 1.

ⓐ $$\left(e^{i\mathcal{L}t}A\right)(\theta)=\left(U_t A\right)(\theta)= A(\theta+\omega t)$$

take the derivative w/respect to time:

$$\left(i\mathcal{L}\,e^{i\mathcal{L}t}A\right)(\theta)=\omega A'(\theta+\omega t)$$

evaluate at $t=0$:

$$\left(i\mathcal{L}A\right)(\theta)=\omega A'(\theta)$$

$$\left(\mathcal{L}A\right)(\theta)=-i\omega A'(\theta)$$

ⓑ $$\left(\mathcal{L}\,\varphi_n\right)(\theta)=-i\omega\,\varphi_n'(\theta)\overset{\text{set}}{=}\lambda_n\,\varphi_n(\theta)$$

$$\varphi_n'(\theta)=i\,\frac{\lambda_n}{\omega}\,\varphi_n(\theta)$$

$$\varphi_n(\theta)=c_n\,e^{i(\lambda_n/\omega)\theta}$$

but $\varphi_n(\theta+2\pi)=\varphi_n(\theta)$

$\therefore\ \dfrac{\lambda_n}{\omega}=n$, where n is an integer

Normalization: $$1=\int_0^{2\pi}\left|\varphi_n(\theta)\right|^2 d\theta$$

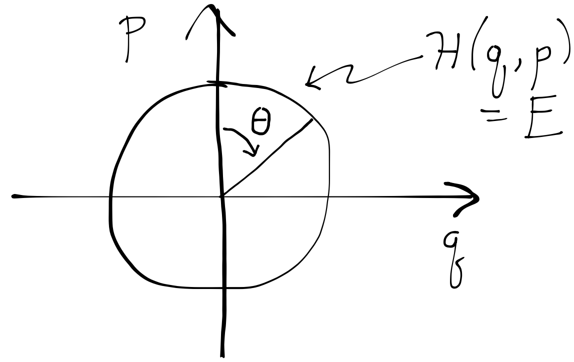
$$\rightarrow\ c_n=\frac{1}{\sqrt{2\pi}}$$

$$\boxed{\varphi_n(\theta)=\frac{1}{\sqrt{2\pi}}\,e^{in\theta}\ ,\quad \lambda_n=n\omega}$$

(c) $p(\theta) = \sqrt{2mE}\,\cos\theta$

$q(\theta) = \sqrt{\dfrac{2E}{m\omega^2}}\,\sin\theta$

$\dot{\theta} = \omega$



(Easy to verify: $\dot{q} = \frac{p}{m}$, $\dot{p} = -m\omega^2 q$)

$$p(\theta) = \sqrt{2mE}\,\cos\theta = \sqrt{\pi m E}\,\varphi_{+1}(\theta) + \sqrt{\pi m E}\,\varphi_{-1}(\theta)$$

$$\text{i.e. } \alpha_n = \begin{cases} \sqrt{\pi m E} & \text{if } n = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

$$(U_t p)(\theta) = e^{i\mathcal{L}t}\alpha\left(\varphi_{+1} + \varphi_{-1}\right) \qquad \alpha \equiv \sqrt{2mE}$$

$$= \alpha\left(e^{i\omega t}\varphi_{+1} + e^{-i\omega t}\varphi_{-1}\right)$$

$$= \sqrt{2mE}\,\cos(\theta + \omega t) = p(\theta + \omega t) \quad \checkmark$$

(d) $$f_0(\theta) = \sum_n \beta_n \varphi_n(\theta)$$

$$\beta_n = \langle n | f_0 \rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\theta'\, e^{-in\theta'} f_0(\theta')$$

$$\therefore\ e^{-i\mathcal{L}t} f_0(\theta) = \sum_n \left[\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\theta'\, e^{-in\theta'} f_0(\theta')\right] e^{-in\omega t} \varphi_n(\theta)$$

$$= \sum_n \frac{1}{2\pi} \int_0^{2\pi} d\theta'\, e^{-in(\theta' + \omega t - \theta)} f_0(\theta)$$

$$= \int_0^{2\pi} d\theta' \left[\frac{1}{2\pi} \sum_n e^{-in(\theta' + \omega t - \theta)} \right] f_0(\theta)$$

$$= \int_0^{2\pi} d\theta' \; \delta(\theta' + \omega t - \theta) \; f_0(\theta)$$

$$= f_0(\theta - \omega t) = f(\theta, t) \quad \checkmark$$

Here we've used the following series representation of the Dirac delta function:

$$\delta(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} e^{inx}$$

Problem 2.

ⓐ $$f(x,t+\delta t) = \int dx_0\, f(x_0,t)\, \mathcal{N} \exp\left[-\frac{(x-x_0+\alpha x_0 \delta t)^2}{4D\delta t}\right]$$

$\epsilon \equiv x - x_0$ $$= \int d\epsilon\, f(x-\epsilon,t)\, \mathcal{N} \exp\left[-\frac{(\epsilon+\alpha(x-\epsilon)\delta t)^2}{4D\,\delta t}\right]$$

$$\epsilon + \alpha(x-\epsilon)\delta t = (1-\alpha\delta t)\epsilon + \alpha x \delta t$$

$$\therefore \frac{(\epsilon+\alpha(x-\epsilon)\delta t)^2}{4D\,\delta t} = \frac{(1-\alpha\delta t)^2}{4D\,\delta t}\left(\epsilon + \frac{\alpha x \delta t}{1-\alpha\delta t}\right)^2$$

$$f(x,t+\delta t) = \int d\epsilon\, f(x-\epsilon,t)\, \frac{1}{\sqrt{4\pi D\delta t}} \exp\left[-\frac{(1-\alpha\delta t)^2}{4D\delta t}\left(\epsilon + \frac{\alpha x\delta t}{1-\alpha\delta t}\right)^2\right]$$

$$= \frac{1}{1-\alpha\delta t}\int d\epsilon\, f(x-\epsilon,t)\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\epsilon+\beta x\delta t)^2}{2\sigma^2}\right]$$

$$\sigma^2 = \frac{2D\delta t}{(1-\alpha\delta t)^2}, \quad \beta = \frac{\alpha}{1-\alpha\delta t}$$

ⓑ $$f(x,t+\delta t) = \frac{1}{1-\alpha\delta t}\int d\epsilon\, f(x-\epsilon,\delta t)\; g(\epsilon)$$

where $g(\epsilon)$ is a Gaussian with mean $-\beta x \delta t$ & variance σ^2

Expand $f(x-\epsilon,t)$ to $\mathcal{O}(\epsilon^2)$ we get the desired result (Eq. 13), with

$$I_0 = \int d\epsilon\, g(\epsilon) = 1$$

$$I_1 = \int d\epsilon\, -\epsilon\, g(\epsilon) = \beta x \delta t$$

$$I_2 = \int d\epsilon \, \frac{\epsilon^2}{2} g(\epsilon) = \frac{\sigma^2}{2} + \frac{(\beta x \delta t)^2}{2}$$

$$= \frac{D \, \delta t}{(1-\alpha \delta t)^2} + \frac{(\beta x \, \delta t)^2}{2}$$

(c) To $O(\delta t)$ we get*

$$f(x, t+\delta t) = f(x,t) + \delta t \left[\alpha f(x,t) + \alpha x f'(x,t) + D f''(x,t) \right]$$

i.e. $$\frac{\partial f}{\partial t} = \alpha f + \alpha x f' + D f'' = \alpha \frac{\partial}{\partial x}(x f) + D \frac{\partial^2 f}{\partial x^2}$$

* using $$\frac{1}{1-\alpha \delta t} = 1 + \alpha \delta t + O(\delta t^2)$$

& $$\beta \delta t = \alpha \delta t + O(\delta t^2)$$

Problem 3

a) Net displacement over K steps:

$$\Delta = \sum_{i=1}^{K} \delta_i = d \sum_{i=1}^{K} x_i \quad , \quad x_i = \pm 1$$

$$\langle \Delta \rangle = d \sum_i x_i = d \cdot K \cdot (p-q)$$

$$\langle \Delta^2 \rangle = d^2 \sum_{i,j} \langle x_i x_j \rangle$$

$$= d^2 \sum_{i=1}^{k} \langle x_i^2 \rangle + d^2 \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} \underbrace{\langle x_i x_j \rangle}_{= \langle x_i \rangle \langle x_j \rangle}$$

$$= K d^2 + K(K-1) d^2 (p-q)^2$$

$$\langle \Delta^2 \rangle - \langle \Delta \rangle^2 = 4pq d^2 K$$

$$2D = \lim_{K \to \infty} \frac{\langle \Delta^2 \rangle - \langle \Delta \rangle^2}{K \epsilon} = \frac{4pqd^2}{\epsilon}$$

$D =$ diff'n coefficient

$v =$ drift coefficient $= \dfrac{\langle \Delta \rangle}{K\epsilon} = \dfrac{d}{\epsilon}(p-q)$

$$\boxed{\frac{\partial f}{\partial t} = -\frac{d}{\epsilon}(p-q) \frac{\partial f}{\partial x} + 2pq \frac{d^2}{\epsilon} \frac{\partial^2 f}{\partial x^2}}$$

$$\to \frac{d^2}{2\epsilon} \frac{\partial^2 f}{\partial x^2} \quad \text{when } p = q = \tfrac{1}{2}$$

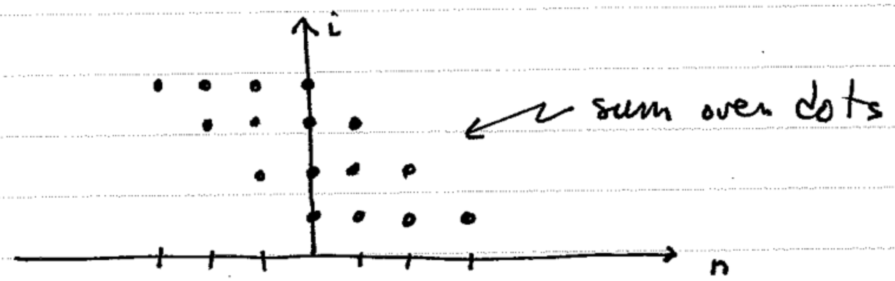
(b) $\Delta = d \sum_i x_i$ as before

$$\Delta^2 = d^2 \sum_{ij} x_i x_j$$

$$\langle \Delta^2 \rangle = d^2 \sum_{ij} \langle x_i x_j \rangle$$

$$= d^2 \sum_{ij} c_{j-i} \qquad c_n \equiv \langle x_0 x_n \rangle$$

$$= d^2 \sum_{i=1}^{K} \sum_{n=1-i}^{K-i} c_n$$



$$\langle \Delta^2 \rangle = d^2 \sum_{n=-K+1}^{K-1} (K - |n|)\, c_n$$

$$= K d^2 \sum_{n=-K+1}^{K-1} \left(1 - \frac{|n|}{K}\right) c_n$$

$$\frac{\langle \Delta^2 \rangle}{K} \xrightarrow[K\to\infty]{} d^2 \sum_{n=-\infty}^{+\infty} c_n \overset{\text{set}}{=} 2 D \epsilon$$

$$D = \frac{d^2}{2\epsilon} \sum_{n=-\infty}^{\infty} c_n$$

$c_0 = \langle x_0^2 \rangle = 1 \qquad c_{-n} = c_n \qquad c_n = \langle x_0 x_n \rangle$

For a given sequence $x_0\, x_1 \cdots x_n$,

$x_0 x_n = (-1)^\ell$ where

$\ell =$ # of "reversals" in the sequence

E.g. for $n=4$: $(+ \quad + \quad + \quad + \quad +) \leftarrow$ no reversals

$(x_0 \quad x_1 \qquad\qquad x_4 \qquad\qquad (\ell=0)$

$(+ \quad - \quad + \quad + \quad -) \leftarrow$ 3 reversals

etc.

ℓ can range from 0 to n,

& the probability to observe ℓ reversals is:

$$\binom{n}{\ell} p^{n-\ell} q^{\ell}$$

$$\therefore \langle x_0 x_n \rangle = \sum_{\ell=0}^{n} \binom{n}{\ell} p^{n-\ell} q^{\ell} \cdot (-1)^\ell = (p-q)^n$$

$$\equiv \delta^n \quad / n \geq 0$$

$$\sum_{n=-\infty}^{+\infty} c_n = \left(\sum_{n=-\infty}^{-1} c_n \right) + c_0 + \left(\sum_{n=1}^{\infty} c_n \right)$$

$$= c_0 + 2 \sum_{n=1}^{\infty} c_n$$

$$= -c_0 + 2 \sum_{n=0}^{\infty} c_n$$

$$= -1 + 2 \sum_{n=0}^{\infty} \left(1 + \delta + \delta^2 + \cdots \right)$$

$$= -1 + \frac{2}{1-\delta} = \frac{1+\delta}{1-\delta} = \frac{p}{q}$$

$$D = \frac{d^2}{2\epsilon} \frac{p}{q}$$

drif term $v=0$ by symmetry

$$\boxed{\frac{\partial f}{\partial t} = \frac{d^2}{2\epsilon} \frac{p}{q} \frac{\partial^2 f}{\partial x^2}}$$

$$\rightarrow \frac{d^2}{2\epsilon} \frac{\partial^2 f}{\partial x^2} \quad \text{when } p = q = \tfrac{1}{2}$$

Problem 4.

(a) $f(p,t) = $ momentum distibution

$$h(\lambda,t) = \int dp\ f(p,t)\, e^{\lambda p} = \langle e^{\lambda p} \rangle$$

$$g(\lambda,t) = \ln h(\lambda,t) \quad \leftarrow \text{generating function}$$

$$\frac{\partial h}{\partial t} = \int dp\ \frac{\partial f}{\partial t} e^{\lambda p} = \int dp\ e^{\lambda p} \frac{\partial}{\partial p}\Big[\frac{\gamma p}{m} f - \underbrace{F(t)}_{qE_0 \cos \omega t} f + D_p \frac{\partial f}{\partial p}\Big]$$

Integrating by parts gives:

$$\frac{\partial h}{\partial t} = -\lambda \int dp\ e^{\lambda p}\Big[\frac{\gamma p}{m} f - F(t) f + D_p \frac{\partial f}{\partial p}\Big]$$

$$= -\lambda \frac{\gamma}{m} \frac{\partial h}{\partial \lambda} + \lambda F(t)\, h + \lambda^2 D_p\, h$$

$$\frac{\partial g}{\partial t} = \frac{1}{h}\frac{\partial h}{\partial t} = -\lambda \frac{\gamma}{m}\frac{\partial g}{\partial \lambda} + \lambda F(t) + \lambda^2 D_p$$

But $g(\lambda,t) = \sum_{l=1}^{\infty} \frac{\kappa_l}{l!}\lambda^l \quad \therefore \quad \frac{\partial g}{\partial t} = \sum_{l=1}^{\infty} \frac{1}{l!}\dot{\kappa}_l \lambda^l$

$$\frac{\partial g}{\partial \lambda} = \sum_{l=1}^{\infty} \frac{1}{(l-1)!} \kappa_l \lambda^{l-1}$$

So: $\dot{\kappa}_1 \lambda + \frac{1}{2}\dot{\kappa}_2 \lambda^2 + \frac{1}{6}\dot{\kappa}_3 \lambda^3 + \cdots$

$$= -\frac{\gamma}{m}\Big(\kappa_1 \lambda + \kappa_2 \lambda^2 + \frac{1}{2}\kappa_3 \lambda^3 + \cdots\Big) + \lambda F(t) + \lambda^2 D_p$$

Collecting terms by power of λ, we get:

$$\boxed{\begin{aligned} \dot{K}_1 &= -\frac{\gamma}{m} K_1 + F(t) \\ \frac{1}{2}\dot{K}_2 &= -\frac{\gamma}{m} K_2 + D_p \\ \frac{1}{n}\dot{K}_n &= -\frac{\gamma}{m} K_n \qquad , \; n \geq 3 \end{aligned}}$$

(b) $n \geq 3$: $K_n(t) = K_n(0)\, e^{-n\nu t} \xrightarrow[t\to\infty]{} 0 \qquad \nu \equiv \frac{\gamma}{m}$

$\langle p \rangle = K_1 = \mu$ (mean of the distribution)

$$\dot{\mu} = -\nu\mu + \alpha \cos\omega t \quad , \quad \alpha \equiv q E_0$$

$$\mu(t) = \mu(0)\, e^{-\nu t} + \int_0^t ds\, \alpha \cos(\omega s)\, e^{-\nu(t-s)}$$

$$= \mu(0)\, e^{-\nu t} + \alpha \,\mathrm{Re}\, \frac{e^{i\omega t} - e^{-\nu t}}{\nu + i\omega}$$

$$\xrightarrow{t\to\infty} \frac{q E_0 m}{\sqrt{\gamma^2 + m^2\omega^2}} \cos(\omega t - \varphi) \quad , \quad \varphi = \tan^{-1}\frac{m\omega}{\gamma}$$

$$\sigma^2 = K_2(t) = \left[K_2(0) - \frac{m D_p}{\gamma}\right] e^{-2\nu t} + \frac{m D_p}{\gamma} \xrightarrow[t\to\infty]{} m\beta^{-1}$$

(c) $F = q E_0 \cos\omega t \quad , \quad v = \frac{p}{m}$

$$\langle \text{power} \rangle = \frac{q E_0}{m} \langle p \rangle \cos\omega t = \frac{q^2 E_0^2}{\sqrt{\gamma^2 + m^2\omega^2}} \cos(\omega t - \varphi) \cos\omega t$$

Averaging over one period $\left(\frac{2\pi}{\omega}\right)$ gives:

$$\overline{\langle \text{power} \rangle} = \frac{q^2 E_0^2}{\sqrt{\gamma^2 + m^2\omega^2}} \frac{1}{2}\cos\varphi = \frac{1}{2} \frac{\gamma q^2 E_0^2}{\gamma^2 + m^2\omega^2}$$

time-averaged dissip'n

(d) $\omega \ll \frac{\gamma}{m}$: $\quad \varphi \to 0 \,, \quad \gamma^2 + m^2\omega^2 \to \gamma^2$

$$\langle p \rangle = \frac{q E_0 m}{\gamma} \cos \omega t$$

$$\overline{\langle \text{power} \rangle} = \frac{q^2 E_0^2}{2\gamma}$$

We get the same results by realizing ~~that~~ for low frequencies the average force due to fiction, $-\frac{\gamma}{m}\langle p \rangle$, is balanced by the external force $q E_0 \cos \omega t$, i.e. the particle is always at "terminal velocity (apart from fluctuations).